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Remarks on the 20-vertex model

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Abstract. A special case of the 20-vertex model on a triangular lattice is solved exactly for all temperatures by the method of Bethe *ansatz*. The model exhibits no phase transition.

1. Introduction

The 20-vertex model on a triangular lattice is an extension of the ice-rule vertex models on square lattice and Kagomé lattice (Lieb and Wu 1972) such that it retains the ice condition. It can be shown that the ice-rule vertex model on a Kagomé lattice is equivalent to a special case of the 20-vertex model (Lin 1976). Baxter (1969) has studied a rotationally-invariant 20-vertex model. Kelland (1974a, b) generalised the result of Baxter and he found that the 20-vertex model can be solved exactly by the method of Bethe *ansatz* if certain conditions among the Boltzmann weights (vertex weights) are satisfied. Unfortunately these conditions are in general temperature dependent‡. The 20-vertex model can be solved exactly by the Pfaffian method (Sacco and Wu 1975) if the Boltzmann weights satisfy certain free-fermion conditions which are again temperature dependent in general. In this paper we solve a special case of the 20-vertex model for all temperatures by the method of Bethe *ansatz*. Our model is not a special case of Kelland's model, nor is it, as formulated, a special case of the model of Sacco and Wu.

2. Definition of the model

Place arrows on the bonds of a triangular lattice so that there are three entering and three leaving each vertex. There are twenty possible vertex configurations. If the configurations with all arrows reversed are identified then we have ten distinct configurations as shown in figure 1. The vertices are associated with the energies e_i. The partition function is

$$Z=\sum\left(\prod \omega_i^{n_i}\right) \tag{1}$$

where $\omega_i=\exp(-e_i/kT)$ are the Boltzmann weights (vertex weights), k is the Boltzmann constant, T is the temperature, n_i is the number of vertices with energy e_i, and the summation is extended to all allowed arrow configurations.

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‡ A special case where these conditions hold at all temperatures was given by Kelland (1974a). In this case the free energy can be expressed in terms of elementary functions (Lin and Wang 1977).

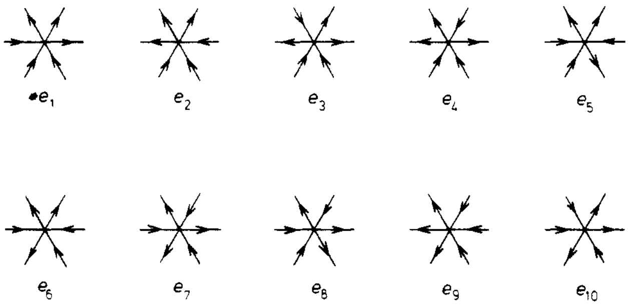

Figure 1. Ten reversal-symmetric vertex configurations and their associated energies.

The partition function possesses some symmetry relations. We write

$$Z = Z(1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \tag{2}$$

where i denotes ω_i. Rotations of the lattice by 60° and 120° lead to

$$Z = Z(2, 9, 6, 8, 4, 10, 7, 3, 1, 5) = Z(9, 1, 10, 3, 8, 5, 7, 6, 2, 4). \tag{3}$$

Reflection symmetry implies

$$Z = Z(2, 1, 6, 5, 4, 3, 7, 10, 9, 8). \tag{4}$$

In this paper, we shall consider the following special case:

$$\begin{aligned} &\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega = \exp(-\epsilon/kT) \\ &\omega_5 = \omega_6 = \omega_7 = \omega_8 = \omega' = \exp(-\epsilon'/kT) \\ &\omega_9 = \omega_{10} = 0. \end{aligned} \tag{5}$$

3. The transfer matrix

Consider a triangular lattice of M rows and each row has N vertices with cyclic boundary conditions as shown in figure 2. We regard the lower (upper) row of non-horizontal bonds as the incoming (outgoing) row. The number n of down arrows in each row is conserved (Baxter 1969) and the transfer matrix is a block diagonal matrix with one block for each value of $n = 0, 1, \ldots, 2N$. The free energy per vertex is

$$F = -kT \lim_{M,N\to\infty} \frac{1}{MN} \ln Z = -kT \lim_{N\to\infty} \frac{1}{N} \ln \Lambda \tag{8}$$

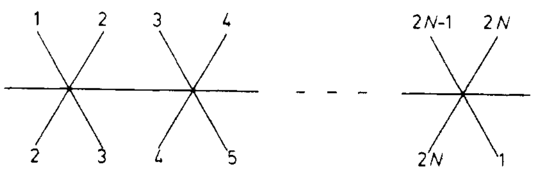

Figure 2. Labelling of the bonds of a triangular lattice.

where Λ is the largest eigenvalue of the transfer matrix. The positions of the down arrows in a row are denoted by $X=\{x_1,\ldots,x_n\}$ and the element of the eigenvector is denoted by $f(X)$ as usual (Lieb and Wu 1972).

In the general case, the transfer matrix is very complicated. It is interesting to note that if $\omega_9=\omega_{10}=0$ then each block matrix (for a given n) is reducible†. To be precise, let us denote by m the number of vertices where two down arrows point out at the same vertex in the lower row. The condition $\omega_9=\omega_{10}=0$ ensures that if m has a certain value in a particular row R, then it cannot be larger than this value in all rows above R. If cyclic boundary conditions are used, then it follows that we need to consider only the subspace with definite values of n and m ($2m\leqslant n$).

4. The Bethe *ansatz*

We now apply the Bethe *ansatz* to solve the model with vertex weights given by equation (5). The largest eigenvalue in the subspace of n and m is denoted by $\Lambda(n, m)$. It is easy to check that $\Lambda(n, m)=\Lambda(n-2m, 0)$ and therefore we only need to consider the case $m=0$. The transfer matrix equation is

$$\Lambda(n,0)f(x_1,\ldots,x_n)=\omega^{N-n}\omega'^{n}\left(\sum_{1}^{x'_1-1}\sum_{x'_1}^{x'_2-1}\cdots\sum_{x'_{n-1}}^{x'_n-1}+\sum_{x'_1}^{x'_2-1}\sum_{x'_2}^{x'_3-1}\cdots\sum_{x'_n}^{2N}\right)f(y_1,\ldots,y_n) \tag{7}$$

where

$$x'=\begin{cases}x & \text{if } x \text{ is even}\\ x-1 & \text{if } x \text{ is odd.}\end{cases}$$

It follows from equation (7) that

$$f(x_1,\ldots,x_n)=f(x'_1,\ldots,x'_n). \tag{8}$$

We use the convention

$$f(x'_1,x'_2,\ldots,x'_n)=f(x'_2,\ldots,x'_n,2N+x'_1). \tag{9}$$

Equation (7) can now be rewritten in the form

$$\Lambda f(x'_1,\ldots,x'_n)=\omega^{N-n}\omega'^{n}\sum_{x'_1}^{x'_2-1}\sum_{x'_2}^{x'_3-1}\cdots\sum_{x'_{n-1}}^{x'_n-1}\sum_{x'_n}^{2N+x'_1-1}f(y'_1,\ldots,y'_n). \tag{10}$$

We try the Bethe *ansatz*

$$f(X)=\sum_P A(P)\exp\left(\mathrm{i}\sum_{j=1}^{n}k_{P(j)}x'_j\right) \tag{11}$$

where the sum is over all permutations $P=\{P(1),P(2),\ldots,P(n)\}$ of the n integers $1,2,\ldots,n$. It is straightforward to show that

$$\{k_j\}=\begin{cases}\pm\dfrac{\pi}{2N},\pm3\dfrac{\pi}{2N},\ldots,\pm(n-1)\dfrac{\pi}{2N} & \text{if } n \text{ is even}\\[2ex] 0,\pm\dfrac{\pi}{N},\pm2\dfrac{\pi}{N},\ldots,\pm\dfrac{n-1}{2}\dfrac{\pi}{N} & \text{if } n \text{ is odd,}\end{cases} \tag{12}$$

† A square matrix is called reducible if there is a permutation of indices which reduces it to the form $\left[\begin{smallmatrix}A & 0\\ C & B\end{smallmatrix}\right]$ where A and B are square matrices.

$$A(P) = \mathrm{i}^{n'/2}\,\mathrm{sgn}(P) \tag{13}$$

$$\Lambda(n, 0) = \begin{cases} \omega^{N-n}\omega'^{n} 2\left(\prod_j \sin|k_j|\right)^{-1} & \text{if } n \text{ is even} \\ \omega^{N-n}\omega'^{n} 4\left(\prod_j{}' \sin|k_j|\right)^{-1} & \text{if } n \text{ is odd} \end{cases} \tag{14}$$

where sgn(P) is the signature of the permutation, and Π' means that we exclude $k_j = 0$. We have $(0 \leqslant r \leqslant 1)$

$$g(r) = \lim_{\substack{n,N\to\infty \\ r=n/N=\text{constant}}} \frac{1}{N} \ln \Lambda(n, 0)$$

$$= -\frac{\epsilon}{kT} - r\frac{\epsilon' - \epsilon}{kT} - \frac{2}{\pi}\int_0^{\frac{1}{2}\pi r} \ln \sin\theta \, \mathrm{d}\theta. \tag{15}$$

For $\epsilon' < \epsilon$ $g(r)$ is a monotonically increasing function of r and the free energy is given by

$$F = -kTg(1) = \epsilon' - kT \ln 2 \qquad \text{if } \epsilon' < \epsilon. \tag{16}$$

For $\epsilon' > \epsilon$ we have

$$\frac{\mathrm{d}g(r)}{\mathrm{d}r} = \frac{\epsilon - \epsilon'}{kT} - \ln\sin(\tfrac{1}{2}\pi r) = 0 \qquad \text{or} \qquad \sin(\tfrac{1}{2}\pi r) = \mathrm{e}^{-(\epsilon'-\epsilon)/kT} \tag{17}$$

and

$$F = \epsilon + r(T)(\epsilon' - \epsilon) + \frac{2}{\pi} kT \int_0^{\frac{1}{2}\pi r(T)} \ln\sin\theta \, \mathrm{d}\theta \qquad \text{if } \epsilon' > \epsilon \tag{18}$$

where $r(T)$ is the solution of equation (17).

Equation (16) implies that for $\epsilon' < \epsilon$, the system is in a 'frozen' state and the free energy is independent of ϵ. This result can be generalised and we have

$$F = -kT \ln 2 + \epsilon'$$

if

$$e_9 = e_{10} = \infty, \qquad \epsilon' = e_5 = e_6 = e_7 = e_8 \leqslant \min\{e_1, e_2, e_3, e_4\}. \tag{19}$$

To see this, notice that

$$Z = \sum\left(\prod \omega_i^{n_i}\right) \leqslant \bar{Z} = \sum\left(\prod \bar{\omega}_i^{n_i}\right) \qquad \text{if } \omega_i \leqslant \bar{\omega}_i \text{ for all } i \tag{20}$$

and therefore we have

$$\bar{\bar{Z}} \leqslant Z \leqslant \bar{Z} \tag{21}$$

where

$$\bar{e}_i(i \leqslant 4) = \min\{e_1, e_2, e_3, e_4\} = e_{\min}$$

$$\bar{\bar{e}}_i(i \leqslant 4) = \max\{e_1, e_2, e_3, e_4\}$$

$$e_i = \bar{e}_i = \bar{\bar{e}}_i \qquad \text{if } i > 4.$$

It follows from inequalities (21) that $Z = \bar{Z} = \bar{\bar{Z}}$ if $e_9 = e_{10} = \infty$ and $e_5 = e_6 = e_7 = e_8 \leqslant e_{\min}$.

Our model can be shown to be in some sense equivalent to a special case of Sacco and Wu (1975)†. Regard their bonds as down- or left-pointing arrows. If $m=0$, then their configurations $\bar{f}_0$, $\bar{f}_{14}$, $\bar{f}_{15}$, $\bar{f}_{16}$ do not occur, so these weights can be set equal to zero. Their Pfaffian conditions (16) and (17) are then satisfied and

$$\begin{aligned} &f_0=f_{14}=f_{15}=f_{16}=\omega \\ &f_{24}=\bar{f}_{24}=f_{25}=\bar{f}_{25}=f_{34}=\bar{f}_{34}=f_{35}=\bar{f}_{35}=\omega'. \end{aligned} \tag{22}$$

All other f_{ij} and $\bar{f}_{ij}$ are zero. Their $D(\theta,\phi)$ in equation (20) is then given by

$$D(\theta,\phi)=|e^{i\phi}-e^{i(\theta+\phi)}+2(\omega'/\omega)|^2. \tag{23}$$

The free energy is therefore given by their equation (20), i.e.

$$-\frac{F}{kT}=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\ln(2\omega'+2\omega\cos\alpha\, e^{i\beta})\,d\alpha\, d\beta \tag{24}$$

where $\alpha+\beta=\phi$ and $\beta-\alpha+\pi=\theta+\phi$. This is rather an elegant formula, true whether $\epsilon'>\epsilon$ or $\epsilon'<\epsilon$. Equations (16) and (18) can be obtained from the above formula by performing the β-integration. Note that equation (13) implies that expression (11) is a determinent, and this is the typical result obtained by applying the Bethe *ansatz* to a problem which can be solved combinatorially by Pfaffians.

5. Conclusion

We have solved exactly a special case of the 20-vertex model on a triangular lattice for all temperatures by the method of Bethe *ansatz*. For $\epsilon>\epsilon'$ the system is in a frozen state for all temperatures while for $\epsilon<\epsilon'$ the model exhibits no phase transition.

The fact that the maximal value of $\Lambda(n,0)$ occurs for $0<r<1$ when $\epsilon'>\epsilon$ implies that our model has a spontaneous partial polarisation. To see this, let us define the vertical polarisation by

$$P=(\text{number of up arrows}-\text{number of down arrows})/2N$$

If $m=0$, P is equal to $1-r$ which is positive. In general for $n\leqslant N/2$ we have $P=1-r-2m/N\geqslant 0$ where $1-r\geqslant 2m/N\geqslant 0$ and $r\leqslant n/N\leqslant\frac{1}{2}$. On the other hand, the model is unchanged by reversing all the arrows, therefore our model has a spontaneous polarisation such that $|P|\leqslant 1-r$.

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† I thank the referee for pointing this out to me.